Comparison of the Bootstrapping Method and Generalized Bootstrapping Method

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Part II

Fitting Data and Distribution with GLD/EGLD

The GLD Moments

The k-th moment of a $GLD(0, \lambda_2, \lambda_3, \lambda_4)$ random variable Z is given by

$$E(Z^k) = \frac{1}{\lambda_2^k} \sum_{i=0}^{k} \left[\binom{k}{i} (-1)^i \beta(\lambda_3(k-i)+1, \lambda_4 i + 1)) \right] \qquad (7)$$

where $\beta(a, b)$ is the beta function defined by

$$\beta(a, b) = \int_0^1 x^{a-1}(1-x)^{b-1} dx.$$

and the k-th moment exists if and only if $\lambda_3 > -1/k$ and $\lambda_4 > -1/k$.

The GLD first four Moments

For a $GLD(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$ r.v. X, its first four moments, α_1,α_2, α_3 and α_4, are given by

$$\alpha_1 = \mu = E(X) = \lambda_1 + \frac{A}{\lambda_2}, \quad (8)$$

$$\alpha_2 = \sigma^2 = E(X-\mu)^2 = \frac{B - A^2}{\lambda_2^2}, \quad (9)$$

$$\alpha_3 = E(X-\mu)^3/\sigma^3 = \frac{C - 3AB + 2A^3}{\lambda_2^3 \sigma^3}, \quad (10)$$

$$\alpha_4 = E(X-\mu)^4/\sigma^4 = \frac{D - 4AC + 6A^2B - 3A^4}{\lambda_2^4 \sigma^4}, \quad (11)$$

Note that $\lambda^2\sigma^2$ in (10) and (11) can be replaced by $B - A^2$.

Estimation via the MoM
Fitting with the EGLD
Estimation via the Percentiles
Estimation via the L-Moments
The GLD Moments
Estimation via the MoM
Numerical Solution
Moment Space

The GLD first four Moments

where

$$A = \frac{1}{1+\lambda_3} - \frac{1}{1+\lambda_4}, \quad (12)$$

$$B = \frac{1}{1+2\lambda_3} + \frac{1}{1+2\lambda_4} - 2\beta(1+\lambda_3, 1+\lambda_4), \quad (13)$$

$$C = \frac{1}{1+3\lambda_3} - \frac{1}{1+3\lambda_4} - 3\beta(1+2\lambda_3, 1+\lambda_4) + 3\beta(1+\lambda_3, 1+2\lambda_4), \quad (14)$$

$$D = \frac{1}{1+4\lambda_3} + \frac{1}{1+4\lambda_4} - 4\beta(1+3\lambda_3, 1+\lambda_4) + 6\beta(1+2\lambda_3, 1+2\lambda_4) - 4\beta(1+\lambda_3, 1+3\lambda_4). \quad (15)$$

Note that A, B, C, D are free of λ_1 and λ_2.

Estimation via the MoM

We can estimate the unknown parameters, λ's, by equating the first four moments to the first four sample moments, $\hat{\alpha}_1$, $\hat{\alpha}_2$,$\hat{\alpha}_3$,$\hat{\alpha}_4$. Since the inequality

$$\alpha_4 > 1 + \alpha_3^2 \tag{16}$$

holds for all distributions, the $GLD(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$ can match any first two moments and some third and fourth moments with (16) holds.
We can first solve the subsystem

$$\alpha_3 = \hat{\alpha}_3 \quad \textit{and} \quad \alpha_4 = \hat{\alpha}_4 \tag{17}$$

for λ_3 and λ_4, then compute λ_1 and λ_2 by (8) and (9).

Estimation via the MoM
Fitting with the EGLD
Estimation via the Percentiles
Estimation via the L-Moments
The GLD Moments
Estimation via the MoM
Numerical Solution
Moment Space

Numerical Solution

However, (17) is complex enough to prevent exact solution. We will find the approximate solutions via numerical method, the Newton's method.
We developed an algorithm to search for a solution near λ_3^* and λ_4^*. The determination is made by checking if

$$\max(|\alpha_3 - \hat{\alpha}_3|, |\alpha_4 - \hat{\alpha}_4|) < \epsilon. \tag{18}$$

If the initial values does not provide a sufficiently accurate solution, the algorithm searches for a better choice of λ_3 and λ_4 and iterates this process until a suitable solution is discovered.
By default, we took ϵ=1.0E-5 and maximum number of iterations 10 in the algorithm.

Bin WANG
Comparison of the Bootstrapping Method and Generalized Boots

Multivariate Newton's Method

Rewrite the subsystem and we have

$$f_1(\lambda_3, \lambda_4) = \alpha_3 - \hat{\alpha}_3 = 0 \quad (19)$$
$$f_2(\lambda_3, \lambda_4) = \alpha_4 - \hat{\alpha}_4 = 0. \quad (20)$$

Define the vector-valued functions $F(\lambda_3, \lambda_4) = (f_1, f_2)$. Thus $F(\lambda) = 0$, where $\lambda = (\lambda_3, \lambda_4)$. The **Jacobian matrix** is defined by

$$DF(\lambda) = \begin{bmatrix} \frac{\partial f_1}{\partial \lambda_3} & \frac{\partial f_1}{\partial \lambda_4} \\ \frac{\partial f_2}{\partial \lambda_3} & \frac{\partial f_2}{\partial \lambda_4} \end{bmatrix}$$

Estimation via the MoM
Fitting with the EGLD
Estimation via the Percentiles
Estimation via the L-Moments
The GLD Moments
Estimation via the MoM
Numerical Solution
Moment Space

Multivariate Newton's Method

The Taylor expansion for vector-valued functions around λ_0 is

$$F(\lambda) = F(\lambda_0) + DF(\lambda_0) \cdot (\lambda - \lambda_0) + O(\lambda - \lambda_0)^2. \quad (21)$$

By ignoring the $O(h^2)$ terms, we can solve (21) for λ,

$$\lambda \approx \lambda_0 - DF^{-1}(\lambda_0) \cdot F(\lambda_0). \quad (22)$$

Multivariate Newton's Method

Iteration Step:

$$\lambda_3^{k+1} = \lambda_3^k - \left.\frac{f_1 \cdot \frac{\partial f_2}{\partial \lambda_4} - f_2 \cdot \frac{\partial f_1}{\partial \lambda_4}}{\frac{\partial f_1}{\partial \lambda_3} \cdot \frac{\partial f_2}{\partial \lambda_4} - \frac{\partial f_1}{\partial \lambda_4} \cdot \frac{\partial f_2}{\partial \lambda_3}}\right|_{\lambda_3=\lambda_3^k,\lambda_4=\lambda_4^k} \quad (23)$$

$$\lambda_4^{k+1} = \lambda_4^k - \left.\frac{f_1 \cdot \frac{\partial f_2}{\partial \lambda_3} - f_2 \cdot \frac{\partial f_1}{\partial \lambda_3}}{\frac{\partial f_1}{\partial \lambda_4} \cdot \frac{\partial f_2}{\partial \lambda_3} - \frac{\partial f_2}{\partial \lambda_4} \cdot \frac{\partial f_1}{\partial \lambda_3}}\right|_{\lambda_3=\lambda_3^k,\lambda_4=\lambda_4^k} \quad (24)$$

Estimation via the MoM
Fitting with the EGLD
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where

$$\frac{\partial f_1}{\partial \lambda_3} = \frac{(C_3' - 3A_3'B - 3AB_3' + 6A^2A_3')(B - A^2)}{(B - A^2)^{5/2}} \frac{-3/2(C - 3AB + 2A^3)(B_3' - 2AA_3')}{(B - A^2)^{5/2}}$$

$$\frac{\partial f_1}{\partial \lambda_4} = \frac{(C_4' - 3A_4'B - 3AB_4' + 6A^2A_4')(B - A^2)}{(B - A^2)^{5/2}} \frac{-3/2(C - 3AB + 2A^3)(B_4' - 2AA_4')}{(B - A^2)^{5/2}}$$

Multivariate Newton's Method

and

$$\frac{\partial f_2}{\partial \lambda_3} = \frac{(D_3' - 4AC_3' - 4A_3'C + 12AA_3'B + 12A^2B_3' - 12A^3A_3')(B - A^2)}{(B - A^2)^3} - \frac{2(D - 4AC + 6A^2B - 3A^4)(B - 2AA_3')}{(B - A^2)^3}$$

$$\frac{\partial f_2}{\partial \lambda_4} = \frac{(D_4' - 4AC_4' - 4A_4'C + 12AA_4'B + 12A^2B_4' - 12A^3A_4')(B - A^2)}{(B - A^2)^3} - \frac{2(D - 4AC + 6A^2B - 3A^4)(B - 2AA_4')}{(B - A^2)^3}$$

Estimation via the MoM
Fitting with the EGLD
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Estimation via the L-Moments
The GLD Moments
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Numerical Solution
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Multivariate Newton's Method

where

$$A_3' = \frac{\partial A}{\partial \lambda_3} = -\frac{1}{(1+\lambda_3)^2}$$

$$A_4' = \frac{1}{(1+\lambda_4)^2}$$

$$B_3' = -\frac{1}{(1+2\lambda_3)^2} - 2\beta_3'(1+\lambda_3, 1+\lambda_4)$$

$$B_4' = -\frac{1}{(1+2\lambda_4)^2} - 2\beta_4'(1+\lambda_3, 1+\lambda_4)$$

$$C_3' = -\frac{1}{(1+3\lambda_3)^2} - 3\beta_3'(1+2\lambda_3, 1+\lambda_4) + 3\beta_3'(1+\lambda_3, 1+2\lambda_4)$$

$$C_4' = \frac{1}{(1+3\lambda_4)^2} - 3\beta_4'(1+2\lambda_3, 1+\lambda_4) + 3\beta_4'(1+\lambda_3, 1+2\lambda_4)$$

Bin WANG
Comparison of the Bootstrapping Method and Generalized Boots

Multivariate Newton's Method

$$
\begin{aligned}
D_3' &= -\frac{1}{(1+4\lambda_3)^2} - 4\beta_3'(1+3\lambda_3, 1+\lambda_4) \\
&\quad +6\beta_3'(1+2\lambda_3, 1+2\lambda_4) - 4\beta_3'(1+\lambda_3, 1+3\lambda_4) \\
D_4' &= -\frac{1}{(1+4\lambda_4)^2} - 4\beta_4'(1+3\lambda_3, 1+\lambda_4) \\
&\quad +6\beta_4'(1+2\lambda_3, 1+2\lambda_4) - 4\beta_4'(1+\lambda_3, 1+3\lambda_4)
\end{aligned}
$$

where

$$
\begin{aligned}
\beta_3'(a+b\lambda_3, c+d\lambda_4) &= \int_0^1 \log(x)\cdot x^{a-1+b\lambda_3}(1-x)^{c-1+d\lambda_4}\,dx \\
\beta_4'(a+b\lambda_3, c+d\lambda_4) &= \int_0^1 \log(1-x)\cdot x^{a-1+b\lambda_3}(1-x)^{c-1+d\lambda_4}\,dx
\end{aligned}
$$

Integration method: Legendre-Gaussian Quadrature

In R, we can use the built-in beta function to compute $\beta(a, b)$:

double beta(*double a, double b*)

$\beta_3'(a+b\lambda_3, c+d\lambda_4)$ and $\beta_4'(a+b\lambda_3, c+d\lambda_4)$ can be computed by using the **Legendre-Gaussian Quadrature**:

$$
\begin{aligned}
\int_0^1 g(x) &= \frac{1}{2}\int_{-1}^1 g\left(\frac{t+1}{2}\right)dt, \quad t = 2x-1 \\
&= \sum_{i=1}^n c_i g(x_i),
\end{aligned}
$$

where c_i and roots x_i are listed below,

Legendre-Gaussian Quadrature Coefficients

n	roots x_i	coefficients c_i
2	$\pm\sqrt{1/3}$	1
3	$\pm\sqrt{3/5}$	$5/9$
	0	$8/9$
4	$\pm\sqrt{\frac{15+2\sqrt{30}}{35}}$	$\frac{18-\sqrt{30}}{36}$
	$\pm\sqrt{\frac{15-2\sqrt{30}}{35}}$	$\frac{18+\sqrt{30}}{36}$
5	0	$128/225$
	$\pm\frac{1}{21}\sqrt{245-14\sqrt{70}}$	$\frac{1}{900}(322+13\sqrt{70})$
	$\pm\frac{1}{21}\sqrt{245+14\sqrt{70}}$	$\frac{1}{900}(322-13\sqrt{70})$

An Alternative Way

The algorithm can be simplified by approximate the partial derivatives in the Jacobian matrix by three-point center-difference approximation

$$\frac{\partial f_j}{\partial \lambda_3} = \frac{f_j(\lambda_3+h,\lambda_4) - f_j(\lambda_3-h,\lambda_4)}{2h} - \frac{h^2}{6} f'''(c_3)$$

$$\frac{\partial f_j}{\partial \lambda_4} = \frac{f_j(\lambda_3,\lambda_4+h) - f_j(\lambda_3,\lambda_4-h)}{2h} - \frac{h^2}{6} f'''(c_4)$$

where $c_k \in (\lambda_k - h, \lambda_k + h)$, $k = 3, 4$.

Limitations of the MoM

- Analyses of actual data shows that it is most common for data to produce (α_3^2, α_4) with
$$\hat{\alpha}_3^2 + 1 \leq \hat{\alpha}_4 \leq 1.8\hat{\alpha}_3^2 + 15,$$
for $-4 \leq \hat{\alpha}_3 \leq 4$ and $1 < \hat{\alpha}_4 \leq 50$.
- The (α_3^2, α_4)-space confined by
$$1.8(\hat{\alpha}_3^2 + 1) \leq \hat{\alpha}_4 \leq 1.8\hat{\alpha}_3^2 + 15$$
is attainable by the GLD.
- The (α_3^2, α_4)-space not attainable by the GLD:
$$\hat{\alpha}_3^2 + 1 \leq \hat{\alpha}_4 \leq 1.8(\hat{\alpha}_3^2 + 1).$$
- Impossible region for all distributions:
$$\hat{\alpha}_4 < \hat{\alpha}_3^2 + 1.$$

Estimation via the MoM
Fitting with the EGLD
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Numerical Solution
Moment Space

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Bin WANG | Comparison of the Bootstrapping Method and Generalized Boots

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Estimation via the MoM
Fitting with the EGLD
Estimation via the Percentiles
Estimation via the L-Moments
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- The (α_3^2, α_4)-space confined by
$$1.8(\hat{\alpha}_3^2 + 1) \le \hat{\alpha}_4 \le 1.8\hat{\alpha}_3^2 + 15$$
is attainable by the GLD.
- The (α_3^2, α_4)-space not attainable by the GLD:
$$\hat{\alpha}_3^2 + 1 \le \hat{\alpha}_4 \le 1.8(\hat{\alpha}_3^2 + 1).$$
- Impossible region for all distributions:
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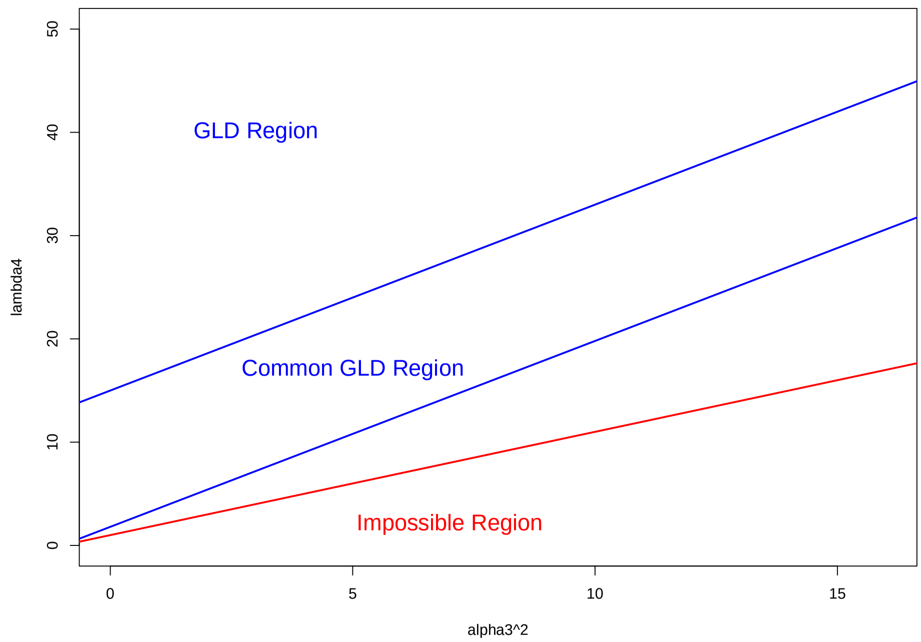
Bin WANG
Comparison of the Bootstrapping Method and Generalized Boots

(α_3^2, α_4)-space



The Generalized β-distribution/EGLD

The EGLD region is not attainable by the method of moments and thus we consider fitting a **Generalized Beta Distribution (GBD)** or **Extended Generalized Lambda Distribution (EGLD)** in that region.

The beta distribution with parameters $\beta_3 > -1$ and $\beta_4 > -1$ is defined through its p.d.f. by

$$f(x) = \begin{cases} \frac{x^{\beta_3}(1-x)^{\beta_4}}{\beta(\beta_3+1,\beta_4+1)}, & \textit{for } 0 \le x \le 1 \\ 0, & \textit{otherwise} \end{cases} \quad (25)$$

where $\beta(a, b)$ is the beta function.

The Generalized β-distribution/EGLD

If $X \sim Beta()$ and $Y = \beta_1 + \beta_2 X$,

$$Y \sim GBD(\beta_1, \beta_2, \beta_3, \beta_4).$$

The p.d.f. of a $GBD(\beta_1, \beta_2, \beta_3, \beta_4)$ r.v. is

$$f(x) = \begin{cases} \frac{(x-\beta_1)^{\beta_3}(\beta_1+\beta_2-x)^{\beta_4}}{\beta(\beta_3+1,\beta_4+1)\beta_2^{\beta_3+\beta_4+1}}, & \text{for } \beta_1 \leq x \leq \beta_1 + \beta_2 \\ 0, & \text{otherwise} \end{cases} \quad (26)$$

The EGLD Moments

The first four moments of a $GBD(\beta_1, \beta_2, \beta_3, \beta_4)$ r.v. are

$$\alpha_1 = \mu'_Y = \beta_1 + \frac{\beta_2(\beta_3+1)}{B_2}, \quad (27)$$

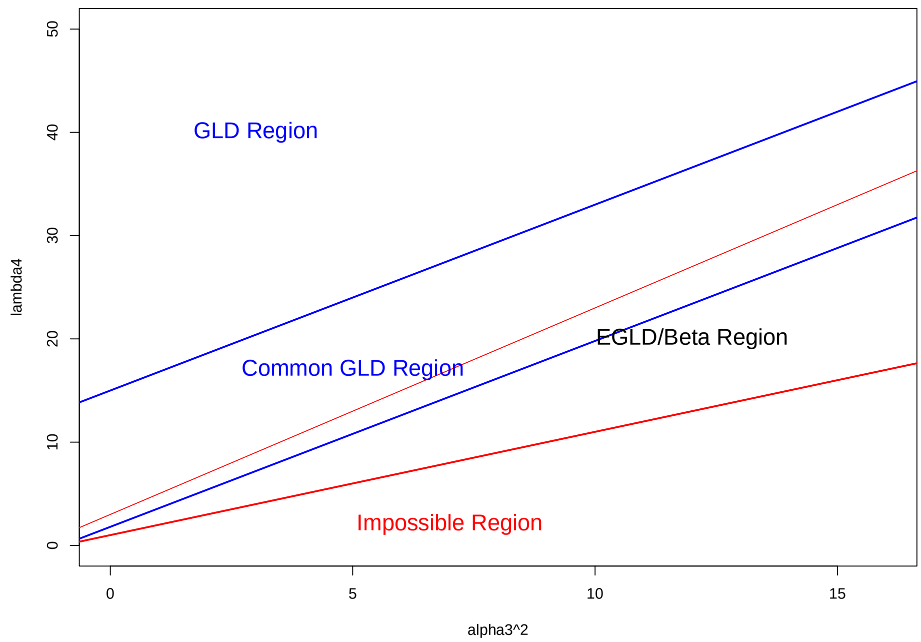
$$\alpha_2 = \mu_2 = \sigma_Y^2 = \frac{\beta_2^2(\beta_3+1)(\beta_4+1)}{B_2^2 B_3}, \quad (28)$$

$$\alpha_3 = \frac{\mu_3}{\sigma_Y^3} = \frac{2(\beta_4-\beta_3)\sqrt{B_3}}{B_4\sqrt{(\beta_3+1)(\beta_4+1)}}, \quad (29)$$

$$\alpha_4 = \frac{\mu_4}{\sigma_Y^4} = \frac{3B_3(\beta_3\beta_4 B_2 + 3\beta_3^2 + 5\beta_3 + 3\beta_4^2 + 5\beta_4 + 4)}{B_4 B_5(\beta_3+1)(\beta_4+1)}, \quad (30)$$

where $B_i = \beta_3 + \beta_4 + i$ for $i = 1, 2, \cdots$.

(α_3^2, α_4)-space



Estimation of GBD Parameters via the MoM

By equating the first four moments above to the first four sample moments, we can estimate the parameters. We can find that the 3rd and 4th moments are free of β_1 and β_2. So, we can solve a subsystem

$$\alpha_i = \hat{\alpha}_i, \quad i = 3, 4,$$

for β_3 and β_4 first. Then use the first two moments to estimate β_2 and β_1.

Numerical solution will be found by using the Newton's method due to the complexity of the subsystem.

Estimation via the Percentiles

Certain moments of some distributions do not exist. The $(100p)$th percentile of a sample $X_1, \cdots, X_n$, $\hat{\pi}_p$, can be obtained from

$$\hat{\pi}_p = X_{(r)} + \frac{a}{b}(X_{(r+1)} - X_{(r)}), \tag{31}$$

where r is the integer part of $(n+1)p$ and a/b is the proper fraction, possibly zero. For $u \in [0, 1/4]$, the sample statistics that we will use are defined by

$$\hat{\rho}_1 = \hat{\pi}_{0.5}, \tag{32}$$
$$\hat{\rho}_2 = \hat{\pi}_{1-u} - \hat{\pi}_u, \tag{33}$$
$$\hat{\rho}_3 = \frac{\hat{\pi}_{0.5} - \hat{\pi}_u}{\hat{\pi}_{1-u} - \hat{\pi}_{0.5}}, \tag{34}$$
$$\hat{\rho}_4 = \frac{\hat{\pi}_{0.75} - \hat{\pi}_{0.25}}{\hat{\rho}_2}, \tag{35}$$

Estimation via the MoM
Fitting with the EGLD
Estimation via the Percentiles
Estimation via the L-Moments
The GLD Percentiles
Estimate the parameters vis a Percentile-based Approach

Estimation via the Percentiles

The $GLD(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$ counterparts are defined as

$$\rho_1 = \lambda_1 + \frac{(1/2)^{\lambda_3} - (1/2)^{\lambda_4}}{\lambda_2}, \tag{36}$$
$$\rho_2 = \frac{(1-u)^{\lambda_3} - u^{\lambda_4} + (1-u)^{\lambda_4} - u^{\lambda_3}}{\lambda_2}, \tag{37}$$
$$\rho_3 = = \frac{(1-u)^{\lambda_4} - u^{\lambda_3} + (1/2)^{\lambda_3} - (1/2)^{\lambda_4}}{(1-u)^{\lambda_3} - u^{\lambda_4} + (1/2)^{\lambda_4} - (1/2)^{\lambda_3}}, \tag{38}$$
$$\rho_4 = \frac{(3/4)^{\lambda_3} - (1/4)^{\lambda_4} + (3/4)^{\lambda_4} - (1/4)^{\lambda_3}}{(1-u)^{\lambda_3} - u^{\lambda_4} + (1-u)^{\lambda_4} - u^{\lambda_3}}. \tag{39}$$

Estimation via a Percentile-based Approach

We can estimate the parameters by solving the system of equations

$$\rho_i = \hat{\rho}_i \qquad \textit{for } i = 1, 2, 3, 4.$$

The last two equations do not depend on λ_1 and λ_2 so we can solve the third and fourth equations for λ_3 and λ_4, then estimate λ_1 and λ_2. Here, we will use numerical search for solutions and take $u = 0.1$.

Probability weighted moments

Greenwood, Landwehr, Matalas and Wallis (1979)
The probability weighted moments of a r.v. X is

$$M_{k,r,s} = E\left[X^k F^r(X)(1 - F(X))^s\right],$$

where $F(X)$ is the distribution function of X. Let

$$\beta_j = M_{1,j,0} = E[XF^j(X)].$$

Then the L-moments of X are defined as

$$\Lambda_1 = \beta_0, \quad \Lambda_i = \sum_{j=0}^{i-1} p_{i,j}\beta_j \quad \textit{for } i = 2, 3, \cdots$$

where

$$p_{i,j} = (-1)^{i-1-j}\binom{i-1}{j}\binom{i+j-1}{j} = \frac{(-1)^{i-1-j}(i+j-1)!}{(j!)^2(i-j-1)!}.$$

The GLD L-moments

The first four L-moments of $GLD(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$ are

$$\Lambda_1 = \lambda_1 + \frac{1}{\lambda_2(\lambda_3+1)} - \frac{1}{\lambda_2(\lambda_4+1)}$$

$$\Lambda_2 = \frac{\lambda_3}{\lambda_2(\lambda_3+1)(\lambda_3+2)} + \frac{\lambda_4}{\lambda_2(\lambda_4+1)(\lambda_4+2)}$$

$$\Lambda_3 = \frac{\lambda_3(\lambda_3-1)}{\lambda_2(\lambda_3+1)(\lambda_3+2)(\lambda_3+3)} - \frac{\lambda_4(\lambda_4-1)}{\lambda_2(\lambda_4+1)(\lambda_4+2)(\lambda_4+3)}$$

$$\Lambda_4 = \frac{\lambda_3(\lambda_3-1)(\lambda_3-2)}{\lambda_2(\lambda_3+1)(\lambda_3+2)(\lambda_3+3)(\lambda_3+4)} - \frac{\lambda_4(\lambda_4-1)(\lambda_4-2)}{\lambda_2(\lambda_4+1)(\lambda_4+2)(\lambda_4+3)(\lambda_4+4)}$$

The Sample L-moments

Let $x_1 \leq x_2 \leq \cdots \leq x_n$ be the order statistics. The sample L-moments can be computed by

$$\ell_i = \sum_{j=0}^{i-1} p_{i,j} b_j,$$

where

$$b_j = \frac{1}{n} \sum_{i=1}^{n} \frac{(i-1)(i-2)\cdots(i-j)}{(n-1)(n-2)\cdots(n-j)} x_i, \quad \textit{for } j = 0, 1, \cdots, n-1.$$

Fitting the GLD via the L-moments

It is easy to show that Λ_3/Λ_2 and λ_4/Λ_2 are free of λ_1 and λ_2. So we can solve a subsystem

$$\frac{\lambda_3}{\Lambda_2} = \frac{\ell_3}{\ell_2} \quad \textit{and} \quad \frac{\lambda_4}{\Lambda_2} = \frac{\ell_4}{\ell_2}$$

for λ_3 and λ_4. Then estimate λ_1 and λ_2 throught the first two L-moments.